OBLIQUE IMPACTS OF METALLIC PLATES IN AN ELASTIC FORMULATION

V. V. Efromov

UDC 539.3

A previous work [1], in which the thickness of the colliding plates was assumed to be infinitely great and the velocity of the point of contact greater than the speed of sound c_1 in the material, represented one attempt to use linear elasticity theory in flash-welding problems. Oblique impact of metallic plates of finite thickness has been studied [2], the velocity V_c of the point of contact being assumed to be less than the velocity c_2 of transverse waves in the material. In this work, oblique impact of elastic plates with velocities V_c of the point of contact greater than the velocity of propagation c_2 of transverse waves will be considered.

Suppose two elastic plates consisting of the same material and having equal thickness h across each other in such a way that their surfaces form an angle γ (impact angle); the plate velocities are directed perpendicular to their surfaces and are equal to v_0. The plates combine into a single plate as a result of the impact, which is in the form of a confluence of two flows in a frame of reference bounded to the point of contact.

The x axis will be directed along the collision angle bisector, while the coordinate origin will be placed at the point of contact. The impact angle will be assumed small, and motion to be steady state. Then the boundary conditions have the following form, taking into account the symmetry of the problem about the x axis:

$$\begin{gathered}\sigma_{yy}=0;\ \sigma_{xy}=0 \text{ when } y=h,\ -\infty<x<\infty;\\ \sigma_{xy}=0 \text{ when } y=0,\ -\infty<x<\infty;\\ v=0 \text{ when } y=0,\ -\infty<x<0;\\ \sigma_{yy}=0 \text{ when } y=0,\ 0<x<\infty;\\ u\to 0;\ v\to -v_0\cos\gamma/2 \text{ when } x\to\infty,\ 0<y<h,\end{gathered} \tag{0.1}$$

where σ_{ik} are the stress-tensor components and u and v are the components of the displacement velocity vector on the x and y axes, respectively.

1. Midsonic Impact Conditions ($c_2 < V_c < c_1$)

As in the case of subsonic impact studied previously, the problem reduces to a Wiener–Hopf equation following a Fourier transformation,

$$b(k)=\frac{2i\lambda_2(\delta-1)\,p(k)\,[\lambda_1\lambda_2\,\mathrm{sh}(k\lambda_2 h)\cos(k\lambda_1 h)-\delta^2}{\left(\delta^4-\lambda_1^2\lambda_2^2\right)\mathrm{sh}(k\lambda_2 h)\sin(k\lambda_1 h)-2\lambda_1\lambda_2\delta^2}\to\frac{\mathrm{ch}(k\lambda_2 h)\sin(k\lambda_1 h)]}{(\mathrm{ch}(k\lambda_2 h)\cos(k\lambda_1 h)-1)}, \tag{1.1}$$

where the functions are given by

$$p(k)=\frac{V_k}{2\mu}\int\limits_{-\infty}^{0}\sigma_{yy}(x,0)\,e^{ikx}dx; \tag{1.2}$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Teknicheskoi Fiziki, No. 5, pp. 167–173, September-October, 1975. Original article submitted October 24, 1974.

$$b(k)=2\int_0^\infty v(x,0)\,e^{ikx}\,dx; \tag{1.3}$$

and the constants by

$$\lambda_1=\sqrt{\left|1-\frac{V_c^2}{c_2^2}\right|},\quad \lambda_2=\sqrt{\left|1-\frac{V_c^2}{c_1^2}\right|},\quad \delta=1-\frac{V_c^2}{2c_2^2},$$

while μ is the shear modulus of the material.

We find, using results of a solution of the motion problem for a load on the surface of an elastic band [3], that elastic waves undamped at infinity cannot exist in front of the point of contact in the case of mid-sonic plate impact conditions, as is the case in subsonic impact conditions [2]. Consequently, the function b(k) is determined by the integral in Eq. (1.3) in the upper half-plane of the complex plane k, except for the point $k = 0$. At the same time, the function p(k) can have infinitely many singular points on the real axis, but is regular in the half-plane $\mathrm{Im}\,k < 0$. If we define b(k), such that it lacks a singular point at the origin, we obtain for the desired function b(k) and p(k) a common pole of regularity, and, consequently, we may solve Eq. (1.1) by the Wiener–Hopf method [4],

$$-\frac{ib(k)\,k\prod_{n=1}^{\infty}\left(1-\frac{k}{2z_n}\right)e^{\frac{k}{2z_n}}\prod_{n=1}^{\infty}\left(1-\frac{k}{2z_n'}\right)e^{\frac{k}{2z_n'}}}{2\lambda_2(\delta-1)\prod_{n=1}^{\infty}\left(1-\frac{k}{z_n}\right)e^{\frac{k}{z_n}}}=\frac{p(k)\,k\prod_{n=1}^{\infty}\left(1-\frac{k}{2z_n}\right)e^{\frac{k}{2z_n}}\prod_{n=1}^{\infty}\left(1-\frac{k}{2z_n'}\right)e^{\frac{k}{2z_n'}}}{\prod_{n=1}^{\infty}\left(1-\frac{k}{z_n}\right)e^{\frac{k}{z_n}}}\times$$

$$\times\frac{\lambda_1\lambda_2\,\mathrm{sh}(k\lambda_2h)\cos(k\lambda_1h)-\delta^2\,\mathrm{ch}(k\lambda_2h)\sin(k\lambda_1h)}{\left(\delta^4-\lambda_1^2\lambda_2^2\right)\mathrm{sh}(k\lambda_2h)\sin(k\lambda_1h)-2\lambda_1\lambda_2\delta^2\left(\mathrm{ch}(k\lambda_2h)\cos(k\lambda_1h)-1\right)}=p_1(k). \tag{1.4}$$

Here $p_1(k)$ is an unknown entire function and z_n and z_n' are, correspondingly, roots of the equations

$$\lambda_1\lambda_2\,\mathrm{th}(k\lambda_2h)=\delta^2\,\mathrm{tg}(k\lambda_1h); \tag{1.5}$$

$$\lambda_1\lambda_2\,\mathrm{tg}(k\lambda_1h)=-\delta^2\mathrm{th}(k\lambda_2h), \tag{1.6}$$

lying in the half-plane k, including the real axis.

Equations (1.5) and (1.6) have only real and purely imaginary roots, the asymptotic of these roots at high |Imz| having the form

$$z_n=-\frac{i\left(\mathrm{arctg}\left(\frac{\delta^2}{\lambda_1\lambda_2}+n\pi\right)\right)}{\lambda_2h}+O\left(e^{-\frac{\lambda_1}{\lambda_2}n\pi}\right);$$
$$z_n'=-\frac{i\left(\frac{\pi}{2}+\mathrm{arctg}\frac{\delta^2}{\lambda_1\lambda_2}+n\pi\right)}{\lambda_2h}+O\left(e^{-\frac{\lambda_1}{\lambda_2}n\pi}\right). \tag{1.7}$$

Consequently, the infinite products in Eq. (1.4) can be replaced to an approximation by the equation

$$\prod_{n=1}^{\infty}\frac{\left(1-\frac{k}{2z_n}\right)e^{\frac{k}{2z_n}}\left(1-\frac{k}{2z_n'}\right)e^{\frac{k}{2z_n'}}}{\left(1-\frac{k}{z_n}\right)e^{\frac{k}{z_n}}}\approx\frac{2^{\frac{2\alpha_0}{\pi}}\,\Gamma\left(\frac{1}{2}+\frac{\alpha_0}{\pi}\right)\Gamma\left(\frac{\alpha_0}{\pi}+m-\frac{ik\lambda_2h}{\pi}\right)}{2^{\frac{ik\lambda_2h}{\pi}+1-m}\sqrt{\pi}\,\Gamma\left(\frac{2\alpha_0}{\pi}+m-\frac{ik\lambda_2h}{\pi}\right)}\times$$

$$\times\prod_{n=0}^{j}\frac{\left(\frac{\pi}{2}+\alpha_0+n\pi\right)(2z_{n+1}-k)\left(2z_{n+1}'-k\right)(\alpha_0+n\pi+m\pi-ik\lambda_2h)}{z_{n+1}'(z_{n+1}-k)\left[2(\alpha_0+n\pi+m\pi)-ik\lambda_2h\right]\left[\pi+2(\alpha_0+n\pi)-ik\lambda_2h\right]}, \tag{1.8}$$

where $\alpha_0=\arctan\delta^2/(\lambda_1\lambda_2)$; $m = 0$ if $\lambda_2^2 < \delta^2$, and $m = 1$ if $\lambda_2^2 > \delta^2$.

We need only set j = 5 in Eq. (1.8) for practical computations in view of the rapid approximation of the roots of Eqs. (1.5) and (1.6) as n increases to their asymptotic form (1.7).

The values of p(k) and b(k) in their domain of definition asymptotically approach zero as |k| increases; consequently, taking into account Eq. (1.8), the entire function $p_1(k)$ has the form

$$p_1(k) = C_1 e^{C_2 k}. \tag{1.9}$$

The constant C_2 is determined from the condition that the stress-tensor components σ_{ik} and deformation velocity vector can have only an integrable singularity at the point of contact [2]

$$C_2 = \frac{i\lambda_2 h \ln 2}{\pi}. \tag{1.10}$$

The constant C_1 is found under the condition that the plate velocities far in front of the point of contact are equal to the throwing speeds (0.1),

$$C_1 = -\frac{v_0 \cos \gamma/2}{\lambda_2(\delta - 1)}. \tag{1.11}$$

Substituting Eqs. (1.8)-(1.11) in Eq. (1.4), we obtain the value of the desired function p(k). The equations relating the stresses and displacements in the material to the Fourier transform of the stresses p(k) acting at the boundary of the elastic band have been presented, in particular, in [2]. The bands of the integrand functions lying on the real axis other than the pole at the point k = 0 must be circuited in the lower half-plane in calculating the inverse Fourier integrals in these equations. Otherwise, the condition that undamped elastic waves cannot exist in front of the point of contact under the given impact conditions will be violated.

The behavior of the stress field near the point of contact can be investigated by studying the asymptotics of the Fourier transforms of the stresses as $|k| \to \infty$ on beams passing through the origin of the complex plane at some nonzero angle to the real axis. In curvilinear coordinates,

$$x = r\cos\varphi,\quad y = \frac{r\sin\varphi}{\lambda_2} \quad (-\pi \leqslant \varphi \leqslant \pi)$$

the equations for the stress-tensor components near the origin have an asymptotic form,

$$\begin{aligned}
\sigma_{xx} &= -\frac{A}{r^{\frac{\alpha_0}{\pi}}}\left\{\frac{\lambda_1\delta}{\left(\cos\varphi + \frac{\lambda_1}{\lambda_2}\sin|\varphi|\right)^{\frac{\alpha_0}{\pi}}} - \frac{\lambda_1(1-\delta+\lambda_2^2)\cos\left(\alpha_0 - \frac{\alpha_0|\varphi|}{\pi}\right)}{\cos\alpha_0}\right\},\\
\sigma_{yy} &= \frac{A}{r^{\frac{\alpha_0}{\pi}}}\left\{\frac{\lambda_1\delta}{\left(\cos\varphi + \frac{\lambda_1}{\lambda_2}\sin|\varphi|\right)^{\frac{\alpha_0}{\pi}}} - \frac{\lambda_1\delta\cos\left(\alpha_0 - \frac{\alpha_0|\varphi|}{\pi}\right)}{\cos\alpha_0}\right\},\\
\sigma_{xy} &= \frac{A\,\mathrm{sign}(\varphi)}{r^{\frac{\alpha_0}{\pi}}}\left\{\frac{\delta^2}{\left(\cos\varphi + \frac{\lambda_1}{\lambda_2}\sin|\varphi|\right)^{\frac{\alpha_0}{\pi}}} - \frac{\delta^2\sin\left(\alpha_0 - \frac{\alpha_0|\varphi|}{\pi}\right)}{\sin\alpha_0}\right\},
\end{aligned} \tag{1.12}$$

if $|\varphi| < \pi - \arctan(\lambda_2/\lambda_1)$;

$$\sigma_{xx} = \frac{A\lambda_1\cos\left(\alpha_0 - \frac{\alpha_0|\varphi|}{\pi}\right)(1-\delta+\lambda_2^2)}{r^{\frac{\alpha_0}{\pi}}\cos\alpha_0},$$

$$\sigma_{yy} = -\frac{A\lambda_1\delta\cos\left(\alpha_0 - \frac{\alpha_0|\varphi|}{\pi}\right)}{r^{\frac{\alpha_0}{\pi}}\cos\alpha_0},\quad \sigma_{xy} = -\frac{A\delta^2\,\mathrm{sign}(\varphi)\sin\left(\alpha_0 - \frac{\alpha_0|\varphi|}{\pi}\right)}{r^{\frac{\alpha_0}{\pi}}\sin\alpha_0},$$

if

$$\pi \geqslant \varphi > \pi - \operatorname{arctg} \frac{\lambda_2}{\lambda_1},$$

where

$$A = -\frac{2\sqrt{\pi}\,\mu \sin\gamma}{\delta(1-\delta)\,\Gamma\left(1-\frac{\alpha_0}{\pi}\right)\Gamma\left(\frac{1}{2}+\frac{\alpha_0}{\pi}\right)}\left(\frac{e^2\lambda_2 h}{4\pi}\right)^{\frac{\alpha_0}{\pi}}.$$

The pressure $p = \sigma_{ii}/3$ is expressed by the equation

$$p = -\frac{2A\lambda_1(1-\delta)\cos\left(\alpha_0 - \frac{\alpha_0|\varphi|}{\pi}\right)\left(1-\frac{4}{3}\frac{c_2^2}{c_1^2}\right)}{r^{\frac{\alpha_0}{\pi}}\cos\alpha_0}.$$

The stresses in Eqs. (1.12) tend to infinity for a velocity of the point of contact V_c of $\sqrt{2c_2}$. This is due to the fact that the passage to the limit as the velocity of the point of contact approaches $\sqrt{2c_2}$ is not uniform relative to the coordinates r, φ. In fact, the potential φ is identically zero when $\delta = 0$, while the stress field is piecewise-constant.

The second singular point of the solution under midsonic conditions arises if

$$\lambda_2^2 = \delta^2. \tag{1.13}$$

In this case, the values of the solution of the steady-state problem increase without bound with distance from the point of contact, which contradicts the conditions at infinity (0.1). The solution of the dynamic motion problem for a load on the surface of an elastic band that has been previously given [3] show that a growth in the stresses and shifts in the material are observed under these impact conditions with time. Consequently, the steady-state formulation of the problem does not have meaning for a velocity of the point of contact satisfying condition (1.13). This resonance phenomenon results from the velocity of the point of contact coinciding with the velocity of propagation of longitudinal elastic waves in the plane-stressed state (thin plate).

We should note the existence of a qualitative distinction between our solution (1.12) and its hydrodynamic analogue [4]. On the one hand, while the shift velocity field in the hydrodynamic formulation has an exponential singularity at a point of contact with exponent $-1/2$, in the elastic formulation the exponent of the singularity is $-[\arctan(\delta_2/\lambda_1\lambda_2)]/\pi$, i.e., it varies as a function of the velocity at the point of contact between $-1/2$ and 0. On the other hand, the deformation and stress rate in the elastic problem have discontinuities on the lines $x \pm \lambda_1 y = 0$, corresponding to Mach lines for a system of elasticity equations in moving coordinates. However, the pressure $p = -\sigma_{ii}/3$ varies continuously relative to the polar angle φ.

2. Supersonic Impact Conditions ($V_c > c_1$)

The system of elasticity equations in moving coordinates is hyperbolic and a solution can be found by the method of characteristics for velocities of the point of contact greater than the velocity of propagation of longitudinal waves in the material. However, the number of waves reflected from the free surface rapidly grows with distance from the point of contact on the x axis, which leads to a substantial increase in the amount of computations. In this work, the Fourier method will be used, since it allows us to obtain equations for the stresses and shifts in the material in an analytic form.

Using a Fourier transformation, we may reduce the solution of our problem to the equation [2, 5]

$$b(k) = \frac{2i\lambda_2(\delta-1)\,p(k)\,[\lambda_1\lambda_2 \sin(k\lambda_2 h)\cos(k\lambda_1 h) + \delta^2 \cos(k\lambda_2 h)\sin(k\lambda_1 h)]}{(\delta^4 - \lambda_1^2\lambda_2^2)\sin(k\lambda_1 h)\sin(k\lambda_2 h) - 2\lambda_1\lambda_2\delta^2(\cos(k\lambda_1 h)\cos(k\lambda_2 h) - 1)}, \tag{2.1}$$

where the functions b(k) and p(k) are determined, as before, by the integrals (1.2) and (1.3).

The magnitude of the shift velocity along the vertical axis is constant and equal to $v_0 \cos(\gamma/2)$ up to the point of contact and vanishes at the interface between the materials by virtue of symmetry. We therefore obtain

$$b(k) = \frac{2v_0 \cos(\gamma/2)}{ik}. \tag{2.2}$$

Substituting Eq. (2.2) in Eq. (2.1), we obtain an equation for the desired function p(k). We may obtain [2] equations for the Fourier transforms of the stresses and shift velocities in the material. Here the inverse Fourier transformation integral defined in the usual fashion will diverge. However, it can be shown from the general theory of Fourier transformations in a complex region [6] that the inverse Fourier transformation in this case, according to the boundary conditions at infinity (0.1), will satisfy the equations

$$f(x) = \frac{e^{-\varepsilon x}}{2\pi} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{e^{-ikx} - 1}{-ik} \Phi^{-}(k - i\varepsilon)\, dk \quad (x<0), \tag{2.3}$$

$$f(x) = \frac{e^{\varepsilon x}}{2\pi} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{e^{-ikx} - 1}{-ik} \Phi^{+}(k + i\varepsilon)\, dk \quad (x>0),$$

where $\Phi(k) = \Phi^{+}(k) + \Phi^{-}(k)$ is the Fourier transform of the function $f(x)$, $\Phi^{-}(k)$ is regular in the lower half-plane, except for the origin, and $\Phi^{+}(k)$ is regular in the half-plane $|\mathrm{Im}\, k| > 0$ and in a neighborhood of the point $k = 0$.

Let us calculate the stress $\sigma_{yy}(x)$ acting at the interface between the materials. Without loss of generality, we may set $\lambda_1/\lambda_2 = m/n$, where m and n are integers. Then the equation for p(k) takes the form

$$p(k) = \frac{2v_0 \cos(\gamma/2)\left[\left(\delta^4 + \lambda_1^2\lambda_2^2\right)\sin(k\lambda_1 h)\sin(k\lambda_2 h) - 2\lambda_1\lambda_2\delta^2\left(\cos(k\lambda_1 h)\cos(k\lambda_2 h) - 1\right)\right]}{k\lambda_1(\delta - 1)(\lambda_1\lambda_2 + \delta^2)\prod_{i=1}^{2(n+m)}\left(e^{\frac{k\lambda_1 h}{m}} - z_i\right)}, \tag{2.4}$$

where the z_i are the roots of the equation

$$z^{2(n+m)} + \frac{\delta^2 - \lambda_1\lambda_2}{\delta^2 + \lambda_1\lambda_2} z^{2m} - \frac{\delta^2 - \lambda_1\lambda_2}{\delta^2 + \lambda_1\lambda_2} z^{2n} - 1 = 0. \tag{2.5}$$

Since the values of the integrand in Eq. (2.3) are calculated at points ε units apart on the real axis, we may decompose the function of Eq. (2.4) in a series in powers of $e^{-k\lambda_1 h/m}$. Since the resulting series is absolutely summable, term-by-term integration may be carried out. Then we obtain for the stress $\sigma_{yy}(x)$ acting at the interface, the equation

$$\sigma_{yy}(x, 0) = -\frac{v_0 \cos(\gamma/2)(\delta^2 + \lambda_1\lambda_2)}{2\lambda_2(\delta - 1)}\left\{\sum_{p=0}^{\xi} b_p + \sum_{p=0}^{\xi-2n} \beta^2 b_p + \sum_{p=0}^{\xi-n-m} 2(\beta^2 - 1) b_p + \sum_{p=0}^{\xi-2m} \beta^2 b_p + \sum_{p=0}^{\xi-2(m+n)} b_p\right\},$$

where

$$\xi = \left[\frac{|x|\, m}{\lambda_1 h}\right]; \quad \beta = \frac{\delta^2 - \lambda_1\lambda_2}{\delta^2 + \lambda_1\lambda_2};$$

and $b_p = \sum_{i_1+\ldots+i_{2(n+m)}=p} z_1^{i_1} z_2^{i_2} \ldots z_{2(n+m)}^{i_{2(n+m)}}$ is the sum of the products of the roots of Eq. (2.5).

The values of the coefficients b_p will be calculated using the recurrence formula

$$b_p = -\sum_{i=1}^{s} b_{p-i} a_i, \quad b_0 = 1, \quad b_1 = -a_1, \tag{2.6}$$

where $s = p$ if $p < 2(n + m)$, $s = 2(n + m)$ if $p > 2(n + m)$, and a_i are the coefficients of the polynomial in Eq. (2.5). In this case, $a_0 = 1$, $a_{2n} = \beta$, $a_{2m} = -\beta$ and $a_{2(n+m)} = -1$. The remaining a_i are zero. Equations (2.6) can be proved by using Vieta's formulas.

The values of the stresses at any point of the material can be obtained by using this method of transforming Fourier integrals.

Our impact model for metallic plates in the elastic formulation does not describe all phenomena occurring in flash-welding problems. However, it allows us to estimate, in particular, the magnitude of the tangential stresses, which cannot be done within framework of hydrodynamic theory [4]. It is possible to calculate the breaking stresses that occur at the interface between materials which exert a substantial influence on the strength of the resulting compound by means of our model. It is apparently possible to theoretically approach a definition of the region of impact conditions in order to obtain high-quality welding for a given pair of materials by using calculations within the elastic model.

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